Descriptive Set Theory
Lecture 11

Theorem. (a) Player 1 has a winning strategy $\Leftrightarrow$ $B \supseteq$ perfect set $\neq \emptyset$.

(b) Player 2 has a winning strategy $\Leftrightarrow$ B is ctbl.

Proof. (b) $\Leftarrow$. Let $B = \{b_n\}_{n \in \mathbb{N}}$. For the $2n^{th}$, when offered disjoint open sets $U_0^{(n)}$ and $U_1^{(n)}$, take the one that doesn't contain b_n.

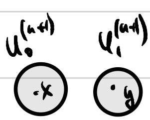
$\Rightarrow$. Now suppose σ is a winning strategy for P2.

We say that a position $p := ((U_0^{(0)}, U_1^{(0)}), i_0, \dots, (U_0^{(n)}, U_1^{(n)}), i_n)$ in σ is good for a point $x \in X$ if $x \in U_{i_n}^{(n)}$, i.e. x is still a possible outcome of the game.

Since σ is a winning strategy for P2, each $b \in B$ has a maximal good position $p \in \sigma$, i.e. $\nexists\, q \supsetneq p$ in σ s.t. q is still good for b. In other words,

$$B \subseteq \bigcup_{\substack{p \in \sigma \\ p \text{ even}}} M_p, \text{ where } M_p := \{x \in X : p \text{ is max-good for } x\}$$

Since σ is ctbl, it remains to show that each M_p is ctbl. But we show that, in fact, $|M_p| \leq 1$. Suppose M_p contains $x \neq y$. Then $\exists\, (U_0^{(n+1)}, U_1^{(n+1)})$ s.t. $U_0^{(n+1)} \ni x$ and $U_1^{(n+1)} \ni y$ and is a legal move for P1.



But then no matter what P2 plays, the resulting position

$p^\frown (U_0^{(n+1)}, U_1^{(n+1)})^\frown i_{n+1}$ is still good for x or y contradicting the maximal goodness of p for both x and y. $\boxed{b}$

Corollary (AD). All sets in Polish spaces have the PSP.

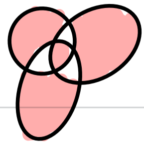
Proof. By restricting to the perfect kernel of a given Polish space X, we may assume WLOG that X is nonempty perfect. But then AD implies that the cut-and-choose game $G^{0/0}(B)$ is determined for any set $B \subseteq X$, hence B has the PSP. $\square$

Baire measurability/property. For an ideal $\mathcal{I}$ on a set X, define an equivalence relation $=_\mathcal{I}$ on $\mathcal{P}(X)$ by:

$$A =_\mathcal{I} B \;:\Longleftrightarrow\; A \triangle B \in \mathcal{I}.$$

This is an eq. rel. because $A \triangle A = \emptyset \in \mathcal{I}$, $\triangle$ is commutative, and "$A =_\mathcal{I} B$ and $B =_\mathcal{I} C \Rightarrow A \triangle C = (A \triangle B) \triangle (B \triangle C) \subseteq (A \triangle B) \cup (B \triangle C) \in \mathcal{I} \Rightarrow A =_\mathcal{I} C$)".

Secret. $\triangle$ operation makes $\mathcal{P}(X)$ into an abelian group with $\emptyset$ as the identity, in which every element has order 2.
Indeed, $A \triangle A = \emptyset$ so the only thing to check is

associativity: $A \Delta (B \Delta C) =$  $= C \Delta (A \Delta B) = (A \Delta B) \Delta C$.

Let $MGR(X)$ be the σ-ideal of meager sets in a top space X.
We denote $=^*$ the eq. rel. $=_{MGR(X)}$ on $\mathcal{P}(X)$.

Def. A subset $A \subseteq X$ is **Baire measurable** (or has the **Baire property**) if $A =^* U$ for some open $U \subseteq X$.

Prop. The collection $BM(X)$ of Baire meas. sets is a σ-algebra containing all open sets, hence all Borel sets.

Proof. Complements: If $A =^* U$ for U open, then $A^c =^* U^c$, so it's enough to show that closed sets are Baire measurable. But for a closed set F, $F = (F \setminus \mathrm{int}(F)) \cup \mathrm{int}(F)$ and since $F \setminus \mathrm{int}(F)$ is nowhere dense, we get $F =^* \mathrm{int}(F)$.

Ctbl-unions: If $A_n =^* U_n$ open, then $\bigcup_n A_n =^* \bigcup_n U_n$ because $\left(\bigcup_n A_n \Delta \bigcup_n U_n\right) \subseteq \bigcup_n (A_n \Delta U_n)$. □

Upgrade property. For any top space X and $A \subseteq X$, TFAE:

(1) A is Baire measurable.

(2) $A = G \cup M$ for some G G_δ and M meager sets.

(3) $A = F \setminus M$ for some F F_σ and M meager sets.

Proof. (1) $\Rightarrow$ (2). $A =^* U$, i.e. $A \triangle U$ is meager $\subseteq \bigcup_n F_n$, where each F_n is closed nowhere dense, so $G := U \setminus \bigcup_n F_n$ and $A = G \cup M$, where $M := (\bigcup_n F_n) \cap A$. □

Corollary. Every nonmeager Baire measurable subset A of a Polish space X contains a nonempty perfect set.

Proof. By the upgrade property, $A = G \cup M$, where G is G_δ and M is meager, so G is nonmeager. Thus, G is an unctbl Polish space $\Rightarrow$ contains a nonempty perfect set. □

Corollary (AC). Every unctbl Polish space has a non-Baire-measurable subset.

Proof. Let $B \subseteq X$ be a Bernstein set, i.e. B and B^c don't contain a nonempty perfect set. But $X = B \cup B^c$, so because X is Baire, one of B or B^c has to be nonmeager, so it is not Baire measurable by the previous corollary. □

Def. Let X, Y be top spaces. A function $f: X \to Y$ is called Baire measurable if the f-preimage of every open set is Baire measurable.

Prop. Let X, Y be top. spaces, Y 2^{nd} ctbl. Any Baire measurable $f: X \to Y$ is continuous on a comeager set, i.e. $\exists$ comeager $X' \subseteq X$ s.t. $f|_{X'}: X' \to Y$ is continuous.

Proof. Fix a ctbl basis (V_n) for Y. It's enough to make sure that $f^{-1}(V_n)$ is open for every $n \in \mathbb{N}$. But $f^{-1}(V_n) =^* U_n$ for some open $U_n \subseteq X$. We through out $f^{-1}(V_n) \triangle U_n$ for X, i.e. let $X' := X \setminus \bigcup_n (f^{-1}(V_n) \triangle U_n)$. Then $(f|_{X'})^{-1}(V_n) = U_n \cap X'$. $\square$

Localization. This means restricting to some/all open sets.

100% lemma. For a Baire meas. set A in a top. space X, if A is nonmeager, then it is comeager in some nonempty open set U, i.e. $A \cap U$ is comeager in U, i.e. $U \setminus A$ is meager in U (equiv. in X).

Proof. $A =^* U$, where U is open. If A is nonmeager, then $U \neq \emptyset$

and $U \setminus A$ is meager. $\square$

We define the forcing relation $U \Vdash B$ for an open set U and a set $B \subseteq X$ to mean that $U \setminus B$ is meager (in X or equiv. in U). In other words, if we think B as the set of points of colour blue, then $U \Vdash B$ says that 100% of U is blue. We read $U \Vdash B$ as U forces B.

Prop. Let X be a top. space and $\mathcal{U}$ an open basis for X.

(a) $U \Vdash \bigcap_n A_n \iff \forall n \; U \Vdash A_n$.

(b) Assume X is Baire.
$$U \Vdash B^c \iff \forall^{\emptyset\neq} V \subseteq U \, (V \not\Vdash B),$$
where $V \neq \emptyset$ and $\in \mathcal{U}$.

(c) Assume X is Baire.
$$U \Vdash \bigcup_n U_n \iff \forall \emptyset \neq V \subseteq U \; \exists \emptyset \neq W \subseteq V \; \exists n \, (W \Vdash A_n)$$

Proof. HW. $\square$